

An inequality for warped product CR -submanifolds in an LCK -space form

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Abstract. In this paper, we obtain a geometric inequality for the length of the second fundamental form in terms of the warping function of a CR -warped product submanifold in a locally conformal Kaehler space form. The inequality is discussed for the important subclass of locally conformal Kaehler manifolds i.e., Vaisman manifold.

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Key words: Locally conformal Kaehler; Vaisman manifold; CR -submanifolds; warped products.

1 Introduction

It is well-known that the notion of warped products plays some important role in differential geometry as well as physics. R.L. Bishop and B. O'Neill in 1969 introduced the concept of a warped product manifold to provide a class of complete Riemannian manifolds with everywhere negative curvature [5]. The warped product scheme was later applied to semi-Riemannian geometry ([1]) and general relativity [2].

Recently, Chen [6] (see also [7]) studied warped product, he considered warped product CR -submanifold in the form $M = N^T \times_f N^\perp$ which is called CR -warped product, where N^T and N^\perp are holomorphic and totally real submanifolds of a Kaehler manifold \tilde{M} . Motivated by Chen's papers many authors studied CR -warped product submanifolds in almost complex as well as contact setting (see [8], [9]). In this paper, we have obtained a general sharp inequality for the length of second fundamental form of CR -warped product submanifolds in a locally conformal Kaehler space form (in short LCK -space form). Also, the inequality is discussed for a Vaisman manifold.

2 Preliminaries

A locally conformally Kaehler (LCK) manifold M is one which is covered by a Kaehler manifold \tilde{M} with the deck transformation group acting conformally on

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\tilde{M} . *LCK* manifolds have been widely studied in the last 30 years (see ([11], [12], [14],[15],[16])). They share some properties with Kaehler manifolds c.f. [13].

An almost Hermitian manifold (\tilde{M}, J, g) is called locally conformally Kaehler (*LCK*) if there exists a closed one-form θ (called the Lee form) such that

$$d\omega = \theta \wedge \omega.$$

Equivalently, any cover \tilde{M} of M on which the pull-back $\tilde{\theta}$ of θ is exact carries a Kaehler form $\Omega = e^{-f}\omega$, where $\tilde{\theta} = df$, and such that $\pi_1(M)$ acts on \tilde{M} by holomorphic homothetics. Conversely, a manifold admitting such a Kaehler covering is necessarily locally conformally Kaehler.

Let \tilde{M} be an *LCK* manifold. Then the vector field λ (the Lee field of \tilde{M}) is defined by $g(X, \lambda) = \theta(X)$. The best known examples of *LCK* manifolds are the Hopf manifolds.

Theorem 2.1 [15] *The almost Hermitian manifold \tilde{M} is an LCK manifold if and only if there is a closed 1-form ω on \tilde{M} such that the Weyl connection be almost complex i.e., $\tilde{\nabla}J = 0$.*

If $\tilde{\nabla}$ denotes the Levi-Civita connection on \tilde{M} , then we have

$$(2.1) \quad (\tilde{\nabla}_U J)V = [\theta(V)U - \omega(V)JU - g(U, V)\mu - \Omega(U, V)\lambda],$$

where $\theta = \omega o J$ and $\mu = -J\lambda$ are the anti-Lee form and the anti-Lee vector field, respectively [15]. In terms of the Lee vector field, above equation can be written as

$$(2.2) \quad (\tilde{\nabla}_U J)V = [g(\lambda, JV)U - g(\lambda, V)JU + g(JU, V)\lambda + g(U, V)J\lambda].$$

The most important subclass of *LCK* manifolds is defined by the parallelism of the Lee form with respect to the Levi-Civita connection of g . Moreover, an *LCK* manifold (\tilde{M}, J, g) is called a Vaisman manifold if $\tilde{\nabla}\theta = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of g ([15], [16]).

An *LCK*-manifold \tilde{M} is called an *LCK*-space form if it has a constant holomorphic sectional curvature c . Then the Riemannian curvature tensor. \tilde{R} of, an *LCK*-space form $\tilde{M}(c)$ with constant holomorphic sectional curvature c is given by is given by Matsumoto [10]

$$(2.3) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) = & \frac{c}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ & + g(JX, W)g(JY, Z)\} - g(JX, Z)g(JY, W) - 2g(JX, Y)g(JZ, X) \\ & + \frac{1}{4}\{P(X, W)g(Y, Z) - P(X, Z)g(Y, W) + g(X, W)P(Y, Z) \\ & - g(X, Z)P(Y, W)\} + \frac{3}{4}\{P(X, JW)g(JY, Z) - P(X, JZ)g(JY, W) \\ & + g(JX, W)P(Y, JZ) - g(JX, Z)P(Y, JW) \end{aligned}$$

$$-2P(X, JY)g(JZ, W) - 2P(Z, JW)g(JX, Y)\},$$

where $\tilde{R}(X, Y, Z, W) = g(\tilde{R}(X, Y)Z, W)$ and

$$(2.4) \quad P(Y, X) = -(\tilde{\nabla}_Y \theta)X - \theta(Y)\theta(X) + \frac{1}{2}\|\theta\|^2 g(X, Y),$$

where $\|\theta\|^2$ denotes the length of the Lee form θ with respect to g .

$$P(X, Y) = P(Y, X), \quad P(X, JY) = -P(JX, Y), \quad P(JX, JY) = P(X, Y).$$

Let $\tilde{M}(J, g, \theta)$ be a complex m -dimensional LCK -manifold and M be a real n -dimensional Riemannian manifold isometrically immersed in \tilde{M} . We denote the metric tensor induced on M by g . Let ∇ be the covariant differentiation with respect to the induced metric on M . Then the Gauss and Weingarten formulas for M are respectively given by

$$(2.5) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N,$$

for any X, Y tangent to M and N normal to M , where ∇^\perp is the connection on the normal bundle $T(M)^\perp$, h is the second fundamental form and A_N is the Weingarten map associated with the vector field $N \in T(M)^\perp$ as

$$(2.6) \quad g(A_N X, Y) = g(h(X, Y), N).$$

The second fundamental form is given by

$$(2.7) \quad ((\tilde{\nabla}_X h)(Y, Z) = \tilde{\nabla}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

for all $X, Y, Z \in T(M)$.

A Riemannian manifold M , isometrically immersed in an LCK -manifold \tilde{M} is called a *CR-submanifold* if there exist on M a differentiable holomorphic distribution D i.e., $JD_x = D_x$, for any $x \in M$ whose orthogonal complement D^\perp in $T(M)$ is totally real on M i.e., $JD_x^\perp \subset T(M)_x^\perp$.

For a CR-submanifold M of an LCK -manifold \tilde{M} , the normal bundle $T(M)^\perp$ is decomposed as

$$T(M)^\perp = FD^\perp \oplus \nu,$$

where ν is the invariant normal subbundle of $T(M)^\perp$ under J . Now, on a CR-submanifold of an LCK -manifold \tilde{M} , we have the following result.

Lemma 2.2 *Let M be a CR-submanifold of an LCK -manifold \tilde{M} . Then we have*

$$(i) \quad g(\nabla_U Z, X) = g(JA_{JZ}U, X) - g(J\lambda, Z)g(JU, X) - g(U, Z)g(\lambda, X) \\ + g(\lambda, Z)g(U, X),$$

$$(ii) \quad A_{JZ}W - A_{JW}Z = g(J\lambda, Z)W - g(J\lambda, W)Z,$$

$$(iii) \quad A_{J\xi}X + A_\xi JX = g(J\lambda, \xi)X - g(\lambda, \xi)JX + g(\lambda, JX)\xi + g(\lambda, X)J\xi,$$

for any $X \in D$, $Z, W \in D^\perp$; $\xi \in \nu$ and $U \in T(M)$.

Proof. The proof is straightforward and may be obtained by using (2.2), (2.5) and (2.6). \square

Let us calculate the holomorphic bisectonal curvature $\tilde{H}_B(X, Z)$ for unit vectors $X \in D$ and $Z \in D^\perp$, where $\tilde{H}_B(X, Z)$ is defined by

$$\tilde{H}_B(X, Z) = \tilde{R}(X, JX; JZ, Z).$$

By the straightforward calculation, we get the following lemma.

Lemma 2.3 *Let \tilde{M} be an LCK-space form and let $X \in D$ and $Z \in D^\perp$ be unit vector fields. Then the holomorphic bisectonal curvature of the plane $X \wedge Z$ is given by*

$$(2.8) \quad \tilde{H}_B(X, Z) = \frac{c}{2} + \frac{1}{2}\{P(X, JX) - P(Z, Z)\}.$$

Proof. By definition, we know that

$$\tilde{H}_B(X, Z) = \tilde{R}(X, JX; JZ, Z).$$

By using equation (2.3) we get

$$\begin{aligned} \tilde{H}_B(X, Z) &= \frac{c}{4}\{2g(JX, Z)^2 + 2\} + \frac{1}{4}\{P(X, JZ)g(JX, Z) - P(JX, Z)g(JX, Z) \\ &\quad + 2P(X, JX) - 2P(Z, Z)\} + \frac{3}{4}\{g(JX, Z)P(JX, Z) - P(X, JZ)g(JX, Z)\}, \end{aligned}$$

for any $X \in D$ and $Z \in D^\perp$ in the plane $X \wedge Z$, from above equation it follows that

$$\tilde{H}_B(X, Z) = \frac{c}{2} + \frac{1}{2}\{P(X, JX) - P(Z, Z)\}.$$

In case of Vaisman manifold, from above lemma we get the following important result.

Corollary 2.4 *Let \tilde{M} be a Vaisman manifold. Then the holomorphic bisectonal curvature of the plane $X \wedge Z$ is given by*

$$(2.9) \quad \tilde{H}_B(X, Z) = \frac{c}{2} + g(Z, \lambda)^2 - g(X, \lambda)g(JX, \lambda).$$

where λ is the Lee vector field.

The proof follows from (2.2) and (2.8). \square

3 CR -warped product submanifolds

Bishop and O'Neill [5] introduced the notion of warped product manifolds. They defined these manifolds as: Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and $f > 0$ a differentiable function on N_1 . Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1 : N_1 \times N_2 \rightarrow N_1$ and $\pi_2 : N_1 \times N_2 \rightarrow N_2$. Then the warped product of N_1 and N_2 denoted by $M = N_1 \times_f N_2$ is a Riemannian manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1*}X, \pi_{1*}Y) + (f \circ \pi_1)^2 g_2(\pi_{2*}X, \pi_{2*}Y)$$

for each $X, Y \in \Gamma(TM)$ and \star is a symbol for the tangent map. Thus we have

$$g = g_1 + f^2 g_2.$$

The function f is called the *warping function* of the warped product [5]. A warped product manifold $N_1 \times_f N_2$ is said to be *trivial* if the warping function f is constant.

We recall the following general result obtained by Bishop and O'Neill [5] for warped product manifolds.

Lemma 3.1 [5] *Let $M = N_1 \times_f N_2$ be a warped product manifold with the warping function f , then for any $X, Y \in T(N_1)$ and $Z, W \in T(N_2)$, we have*

- (i) $\nabla_X Y \in T(N_1)$,
- (ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$,
- (iii) $\nabla_Z W = \nabla_Z^{N_2} W - g(Z, W) \nabla \ln f$,

where ∇ and ∇^{N_2} denote the Levi-Civita connections on M and N_2 , respectively and $\nabla \ln f$ is the gradient of the function $\ln f$.

Lemma 3.2 *If $M = N^T \times_f N^\perp$ is a CR -warped product in an LCK-manifold \tilde{M} . Then*

$$g(h(X, Y), JZ) = g(J\lambda, Z)g(X, Y)$$

$$g(h(JX, W), JZ) = (X \ln f + g(J\lambda, JX))g(Z, W)$$

for any X, Y tangent to N^T and Z, W tangent to N^\perp .

Proof. The first part of this lemma is proved in [4] (see Proposition 3.1). For the second part, by Lemma 3.1 (ii), we have

$$\nabla_X Z = (Z \ln f)X$$

for any $X \in T(N^T)$ and $Z \in T(N^\perp)$. Then from (2.2), we get

$$\begin{aligned} g(h(JX, W), JZ) &= g(\lambda, JZ)g(W, JX) + g(JZ, W)g(\lambda, JX) \\ &\quad + g(W, Z)g(J\lambda, JX) + (X \ln f)g(W, Z). \end{aligned}$$

Thus, the result follows from the above equation. \square

According to Proposition 3.1 in [4] and Theorem 3.5 of [8], the necessary and sufficient condition for a CR -submanifold M of an LCK -manifold \tilde{M} , be a CR -warped product is that Lee-vector field λ is orthogonal to D^\perp and

$$(3.1) \quad A_{JZ}X = g(J\lambda, JX) - (g(J\lambda, X) + JX(\mu))Z.$$

for some smooth function μ on M satisfying $W\mu = 0$ for all $X \in D, Z, W \in D^\perp$.

Let M be a (pseudo-)Riemannian k -manifold with inner product g and e_1, \dots, e_k be an orthonormal frame fields on M . For differentiable function ϕ on M , the gradient $\nabla\phi$ and the Laplacian $\Delta\phi$ of ϕ are defined respectively by

$$g(\nabla\phi, X) = X(\phi),$$

$$(3.2) \quad \Delta\phi = \sum_{j=1}^k \{(\nabla_{e_j} e_j)\phi - e_j e_j(\phi)\} = -\text{div} \nabla\phi$$

for vector field X tangent to M , where ∇ is the Riemannian connection on M . As a consequence, we have

$$(3.3) \quad \|\nabla\phi\|^2 = \sum_{j=1}^k (e_j(\phi))^2.$$

Using the above results, we will prove our main theorem.

Theorem 3.3 *Let $M = N^T \times_f N^\perp$ be a CR -warped product submanifold of an LCK -space form $\tilde{M}(c)$. Then the second fundamental form of M satisfies the following inequality*

$$(3.4) \quad \|h\|^2 \geq \frac{ckp}{2} - p(\Delta \ln f) - p\|\nabla \ln f\|^2 - pk(\lambda \ln f)$$

where $\dim N^T = k$, $\dim N^\perp = p$ and λ is the Lee vector field orthogonal to D^\perp in M .

Proof. We have

$$(3.5) \quad \|h(D, D^\perp)\|^2 = \sum_{j=1}^k \sum_{i=1}^p \|h(X_j, Z_i)\|^2,$$

where X_j for $\{j = 1, \dots, k\}$ and Z_α for $\alpha = \{1, \dots, p\}$ are orthonormal frames on N^T and N^\perp , respectively. On N^T we will consider a local orthonormal frame, namely $\{e_j, J e_j\}$, where $\{j = 1, \dots, k\}$. We have to evaluate $\|h(X, Z)\|^2$ with $X \in D$ and $Z \in D^\perp$. The second fundamental form $h(X, Z)$ is normal to M so, it splits into two orthogonal components

$$(3.6) \quad h(X, Z) = h_{JD^\perp}(X, Z) + h_\nu(X, Z),$$

where $h_{JD^\perp}(X, Z) \in JD^\perp$ and $h_\nu(X, Z) \in \nu$. So

$$(3.7) \quad \|h(X, Z)\|^2 = \|h_{JD^\perp}(X, Z)\|^2 + \|h_\nu(X, Z)\|^2.$$

let's first compute the norm of the JD^\perp -component of $h(X, Z)$. We have

$$(3.8) \quad \|h_{JD^\perp}(X, Z)\|^2 = g(h_{JD^\perp}(X, Z), h(X, Z)),$$

which becomes

$$(3.9) \quad \|h_{JD^\perp}(D, D^\perp)\|^2 = \sum_{j=1}^k \sum_{i=1}^p \{\|h_{JD^\perp}(e_j, Z_i)\|^2 + \|h_{JD^\perp}(Je_j, Z_i)\|^2\}.$$

Using (3.3) and Lemma 3.2, after the computations, we can conclude that

$$(3.10) \quad \|h_{JD^\perp}(D, D^\perp)\|^2 = \|\nabla \ln f\|^2 p + \sum_{j=1}^k \{g(J\lambda, e_j)^2 + g(\lambda, e_j)^2\} \\ + 2p \sum_{j=1}^k \{(Je_j \ln f)g(J\lambda, e_j) - (e_j \ln f)g(\lambda, e_j)\}.$$

Now we will compute the norm of the ν -component of $h(X, Z)$. We have

$$\|h_\nu(X, Z)\|^2 = g(h_\nu(X, Z), h(X, Z)) = g(A_{h_\nu(X, Z)}X, Z).$$

Using (3.1), Lemma 3.2 and the fact that $Jh_{JD^\perp}(X, Z)$ belongs to D^\perp , we obtain

$$(3.11) \quad \|h_\nu(X, Z)\|^2 = g(Jh(X, Z), h(JX, Z)) \\ + \{g(\lambda, X) + g(J\lambda, X)\}\{(JX \ln f) - g(J\lambda, X)\}\|Z\|^2,$$

for any $X \in D$ and $Z \in D^\perp$. Consider the tensor field \tilde{H}_B . As we already have seen

$$(3.12) \quad \tilde{H}_B(X, Z) = g((\tilde{\nabla}_{JX})h(X, Z) - (\tilde{\nabla}_X h)(JX, Z), JZ),$$

for any $X \in D$ and $Z \in D^\perp$. Using the definition of $\tilde{\nabla}h$, we obtain

$$(3.13) \quad \tilde{H}_B(X, Z) = g(\nabla^\perp_{JX} h(X, Z) - h(\nabla_{JX} X, Z) - h(X, \nabla_{JX} Z), JZ) \\ - g(\nabla^\perp_X h(JX, Z) + h(\nabla_X JX, Z) + h(JX, \nabla_X Z), JZ),$$

In order to solve easily, we separate each term

$$(3.14) \quad T_1 = g(\nabla^\perp_{JX} h(X, Z), JZ), \quad T_2 = -g(h(\nabla_{JX} X, Z), JZ), \\ T_3 = -g(h(X, \nabla_{JX} Z), JZ), \quad T_4 = -g(\nabla^\perp_X h(JX, Z), JZ)$$

$$(3.15) \quad T_5 = g(h(\nabla_X JX, Z), JZ), \quad T_6 = g(h(JX, \nabla_X Z), JZ),$$

First we will compute T_1 and T_4

$$(3.16) \quad T_1 = \{-(JX)(JX \ln f) + (JX \ln f)^2\} \|Z\|^2 + g(Jh(X, Z), h(JX, Z)).$$

Similarly

$$(3.17) \quad T_4 = \{-(X)(X \ln f) + (X \ln f)^2\} \|Z\|^2 + g(Jh(X, Z), h(JX, Z)).$$

Then, it is not difficult to show that we have

$$(3.18) \quad T_2 = \{J\nabla_{JX}(X \ln f) - g(J\lambda, \nabla_{JX} X)\} \|Z\|^2$$

and

$$(3.19) \quad T_5 = \{g(J\lambda, \nabla_X JX) - J\nabla_X(JX \ln f)\} \|Z\|^2$$

We direct our attention to the third and sixth terms:

$$(3.20) \quad T_3 = \{(JX \ln f)^2 - g(J\lambda, X)(JX \ln f)\} \|Z\|^2$$

$$(3.21) \quad T_6 = \{(X \ln f)^2 - g(J\lambda, JX)(X \ln f)\} \|Z\|^2.$$

After using all above expressions, equation (3.13) becomes

$$(3.22) \quad \begin{aligned} \tilde{H}_B(X, Z) = & \|Z\|^2 \{-JX(JX \ln f) + (JX \ln f)^2 - X(X \ln f) \\ & + (X \ln f)^2 + J\nabla_{JX}(X \ln f) - g(J\lambda, \nabla_{JX} X) \\ & + g(J\lambda, \nabla_X JX) - J\nabla_X(JX \ln f) + (JX \ln f)^2 \\ & - g(J\lambda, X)(JX \ln f) + (X \ln f)^2 - g(J\lambda, JX)(X \ln f)\} \\ & + 2g(Jh(X, Z), h(JX, Z)). \end{aligned}$$

We can easily prove that

$$(3.23) \quad \begin{aligned} J\nabla_{JX}(X \ln f) = & \nabla_{JX}(JX \ln f) - (JX \ln f)g(\lambda, JX) - (X \ln f)g(\lambda, X) \\ & + (\lambda \ln f)\|X\|^2 \end{aligned}$$

and

$$(3.24) \quad \begin{aligned} J\nabla_X(JX \ln f) = & -\nabla_X(X \ln f) + (JX \ln f)g(\lambda, JX) + (X \ln f)g(\lambda, X) \\ & - (\lambda \ln f)\|X\|^2. \end{aligned}$$

Using (3.23), (3.24) in (3.22), we get

$$(3.25) \quad \begin{aligned} \tilde{H}_B(X, Z) = & \{(\nabla_{JX} JX - (JX)^2) \ln f + (\nabla_X X - (X)^2) \ln f + 2(X \ln f)^2 \\ & + 2(JX \ln f)^2 - g(J\lambda, \nabla_{JX} X) + g(J\lambda, \nabla_X JX) \end{aligned}$$

$$\begin{aligned}
& -3g(\lambda, X)X \ln f + g(J\lambda, X)JX \ln f + \|X\|^2 \lambda \ln f \\
& -2g(JX, X)J\lambda \ln f\} \|Z\|^2 + 2g(Jh(X, Z), h(JX, Z)).
\end{aligned}$$

Using orthonormal frames, we have

$$\begin{aligned}
(3.26) \quad \tilde{H}_B(e_j, Z_i) &= \{((\nabla_{J e_j} J e_j) - (J e_j)^2) \ln f + (\nabla_{e_j} e_j - (e_j)^2) \ln f \\
&+ 2(e_j \ln f)^2 + 2(J e_j \ln f)^2 - g(J\lambda, \nabla_{J e_j} e_j) \\
&+ g(J\lambda, \nabla_{e_j} J e_j) - 3g(\lambda, e_j) e_j \ln f + g(J\lambda, e_j) J e_j \ln f \\
&+ \|e_j\|^2 \lambda \ln f - 2g(J e_j, e_j) J \lambda \ln f\} \|Z_i\|^2 + 2\|h_\nu(e_j, Z_i)\|^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(3.27) \quad \tilde{H}_B(J e_j, Z_i) &= \{((\nabla_{e_j} e_j) - (e_j)^2) \ln f + (\nabla_{J e_j} J e_j - (J e_j)^2) \ln f \\
&+ 2(J e_j \ln f)^2 + 2(e_j \ln f)^2 + g(J\lambda, \nabla_{e_j} J e_j) \\
&- g(J\lambda, \nabla_{J e_j} e_j) - 3g(\lambda, J e_j) J e_j \ln f \\
&- g(J\lambda, J e_j) e_j \ln f + 2g(e_j, J e_j) J \lambda \ln f\} \|Z_i\|^2 \\
&+ \|e_j\|^2 \lambda \ln f + 2\|h_\nu(J e_j, Z_i)\|^2.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
(3.28) \quad \Delta(\ln f) &= \sum_{j=1}^k \{(\nabla_{e_j} e_j)(\ln f) - e_j^2(\ln f)\} \\
&+ \sum_{j=1}^k \{(\nabla_{\phi e_j} \phi e_j)(\ln f) - \phi e_j^2(\ln f)\}.
\end{aligned}$$

Using (3.3), we get

$$(3.29) \quad 2\|\nabla \ln f\|^2 = 2 \sum_{j=1}^k (e_j(\ln f))^2 + 2 \sum_{j=1}^k (\phi e_j(\ln f))^2.$$

Taking the sum of (3.26) and (3.27) and using (3.28) and (3.29), we get

$$\begin{aligned}
(3.30) \quad & 2 \sum_{j=1}^k \sum_{i=1}^p \{\|h_\nu(e_j, Z_i)\|^2 + \|h_\nu(J e_j, Z_i)\|^2\} \\
&= \sum_{j=1}^k \sum_{i=1}^p \{\tilde{H}_B(e_j, Z_i) + \tilde{H}_B(J e_j, Z_i)\} - 2p(\Delta \ln f) \\
&- 4p\|\nabla \ln f\|^2 + 2g(J\lambda, \nabla_{J e_j} e_j)p - 2p\|e_j\|^2 \lambda \ln f \\
&- 2pg(J\lambda, \nabla_{e_j} J e_j) + 4pg(\lambda, e_j) e_j \ln f
\end{aligned}$$

$$+ 4pg(J\lambda, e_j)Je_j \ln f + 4pg(e_j, Je_j)J\lambda \ln f.$$

Now, from (3.30) and (3.9), we get

$$(3.31) \quad \begin{aligned} \|h\|^2 &= \frac{ckp}{2} - p(\triangle \ln f) - p\|\nabla \ln f\|^2 - pk(\lambda \ln f) \\ &+ \sum_{j=1}^k \sum_{i=1}^p \left\{ \frac{1}{2}(P(e_j, Je_j) - P(Z_i, Z_i))p + (g(J\lambda, e_j))^2 \right. \\ &+ (g(\lambda, e_j))^2 + g(J\lambda, \nabla_{Je_j} e_j)p - g(J\lambda, \nabla_{e_j} Je_j)p \\ &\left. + (g(\lambda, e_j)e_j + 4g(J\lambda, e_j)Je_j + 2g(e_j, Je_j)J\lambda)p(\ln f) \right\}. \end{aligned}$$

Hence, the inequality (3.4) follows from (3.31). \square

Corollary 3.4 *Let $M = N^T \times_f N^\perp$ be a CR-warped product of a Vaisman space form $\tilde{M}(c)$. Then the second fundamental form of M satisfies the following inequality*

$$(3.32) \quad \|h\|^2 \geq \frac{ckp}{2} + \frac{p+k}{2}\|\theta\|^2 - p(\triangle \ln f) - p\|\nabla \ln f\|^2 - pk(\lambda \ln f)$$

where $\|\theta\|^2$ is the length of Lee form with respect to g .

Proof. From equation (3.31) and (2.4), we get

$$(3.33) \quad \begin{aligned} \|h\|^2 &= \frac{ckp}{2} - p(\triangle \ln f) - p\|\nabla \ln f\|^2 - pk(\lambda \ln f) \\ &+ \frac{p+k}{2}\|\theta\|^2 + \sum_{j=1}^k \left\{ -g(e_j, \lambda)g(Je_j, \lambda) + (g(J\lambda, e_j))^2 \right. \\ &+ (g(\lambda, e_j))^2 + g(J\lambda, \nabla_{Je_j} e_j)p - g(J\lambda, \nabla_{e_j} Je_j)p \\ &\left. + (g(\lambda, e_j)e_j + 4g(J\lambda, e_j)Je_j + 2g(e_j, Je_j)J\lambda)p(\ln f) \right\} \end{aligned}$$

The inequality (3.32) follows from the above equation. \square

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